# Regularity Properties for a System of Interacting Bessel Processes

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## Abstract

We study the regularity of a diffusion on a simplex with singular drift and reflecting boundary condition which describes a finite system of particles on an interval with Coulomb interaction and reflection between nearest neighbors.

As our main result we establish the Feller property for the process in both cases of repulsion and attraction. In particular the system can be started from any initial state, including multiple point configurations. Moreover we show that the process is a Euclidean semi-martingale if and only if the interaction is repulsive. Hence, contrary to classical results about reflecting Brownian motion in smooth domains, in the attractive regime a construction via a system of Skorokhod SDEs is impossible. Finally, we establish exponential heat kernel gradient estimates in the repulsive regime.

The main proof for the attractive case is based on potential theory in Sobolev spaces with Muckenhoupt weights.

## 1 Introduction and Main Results

The study of Brownian motion or more general diffusions in Euclidean domains with reflecting boundary condition is a classical subject in stochastic analysis, with strong connection to boundary regularity theory for parabolic PDE. Starting from the early works by e.g. Fukushima [12] and Tanaka [20], the field has seen perpetual research activity, c.f. [5, 16] (and e.g. [4] for a more comprehensive list of references). Typically the reflecting process can be obtained in two ways. Either by solving a system of Skorokhod SDE involving the local time at the boundary or via Dirichlet form methods, and under suitable smoothness assumptions on the domain and the coefficients both approaches are equivalent.

In this paper we study a very specific singular case, in which the drift coefficients of the operator may diverge and where the equivalence of the two approaches breaks down but the process exhibits good spatial regularity nevertheless. Our case corresponds to a Dirichlet form obtained from the closure of the quadratic form

$$\mathcal{E}(f,f) = \int_\Omega |\nabla f|^2 q(dx), \quad f \in C^1(\overline{\Omega})$$

on $L^2(\Omega, dq)$, for a very specific choice of domain $\Omega \subset \mathbb{R}^N$ and measure $q(dx)$ (see below). Similar variants were studied under the smoothness assumption $q \in H^{1,1}(\Omega)$ and $\nabla(\log q) \in L^p(\Omega, dq)$, $p > N$ in [22] and [11] respectively, where a Skorokhod decomposition for the induced process still holds.

The case treated in this paper is obtained by choosing

$$\Omega = \Sigma_N = \{x \in \mathbb{R}^N : 0 < x^1 < x^2 \cdots < x^N < 1\}$$

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and

$$q_N(dx) = \frac{1}{Z_\beta} \prod_{i=0}^{N} (x^{i+1} - x^i)^{\frac{\beta}{N+1}-1} dx^1 \, dx^2 \cdots dx^N, \tag{1.1}$$

where by convention $x^0 = 1$, $x^{N+1} = 1$, $Z_\beta = (\Gamma(\beta)/(\Gamma(\beta/(N+1))))^{N+1})^{-1}$ is a normalization constant and $\beta > 0$ is a free parameter.

We study the process $(X_\cdot^N)$ generated by (the $L^2(\Sigma_N, q_N)$-closure of) the (pre-)Dirichlet form

$$\mathcal{E}^N(f, f) = \int_{\Sigma_N} |\nabla f|^2(x) \, q_N(dx), \quad f \in C^\infty(\overline{\Sigma}_N),$$

whose generator $\mathcal{L}$ extends the operator $(L^N, D(L^N))$

$$L^N f(x) = (\frac{\beta}{N+1} - 1) \sum_{i=1}^{N} \left( \frac{1}{x^i - x^{i-1}} - \frac{1}{x^{i+1} - x^i} \right) \frac{\partial}{\partial x^i} f(x) + \Delta f(x) \quad \text{for } x \in \Sigma_N \tag{1.2}$$

with domain

$$D(L^N) = C^2_{Neu} = \{f \in C^2(\overline{\Sigma}_N) \,|\, \nabla f \cdot \nu = 0 \text{ on all } (n-1)\text{-dimensional faces of } \partial\Sigma_N\},$$

and $\nu$ denoting the outward normal field on $\partial\Sigma_N$.

On the level of formal Itô calculus $(L^N, D(L^N))$ corresponds to an order preserving dynamics for the location of $N$ particles in the unit interval which solves the system of coupled Skorokhod SDEs

$$dx_t^i = (\frac{\beta}{N+1} - 1) \left( \frac{1}{x_t^i - x_t^{i-1}} - \frac{1}{x_t^{i+1} - x_t^i} \right) dt + \sqrt{2} dw_t^i + dl_t^{i-1} - dl_t^i, \quad i = 1, \cdots, N \tag{1.3}$$

where $\{w_i\}$ are independent real Brownian motions and $\{l^i\}$ are the collision local times, i.e. satisfying

$$dl_t^i \geq 0, \quad l_t^i = \int_0^t \mathbb{1}_{\{x_s^i = x_s^{i+1}\}} dl_s^i. \tag{1.4}$$

$(X_\cdot^N)$ may thus be considered as a system of coupled two sided real Bessel processes with uniform Bessel dimension $\delta = \beta/(N+1)$. Similar to the standard real Bessel process $BES(\delta)$ with Bessel dimension $\delta < 1$, the existence of $X^N$, even with initial condition initial condition $X_0 = x \in \Sigma_N$, is not trivial, nor are its regularity properties.

Our motivation for studying this process is its relation to the Wasserstein diffusion, c.f. [24]. In [3] we showed that the normalized empirical measure of the system (1.3) converges to the Wasserstein diffusion in the high density regime for $N \to \infty$. Hence the regularity properties of $(X_\cdot^N)$ may give an indication of the regularity of the Wasserstein diffusion, although in the present paper we treat the case when the dimension $N$ is fixed. Here our results read as follows.

**Theorem 1.1.** *For any $\beta > 0$, the Dirichlet form $\mathcal{E}^N$ generates a Feller process $(X_\cdot)$ on $\overline{\Sigma}_N$, i.e. the associated transition semigroup on $L^2(\Sigma_N, q_N)$ defines a strongly continuous contraction semigroup on the subspace $C(\overline{\Sigma}_N)$ equipped with the sup-norm topology.*
*Moreover, for $\beta \geq (N+1)$ the associated heat kernel $(P_t)$ is exponentially smoothing on Lipschitz functions, i.e. for $t > 0$*

$$\text{Lip}(P_t f) \leq \exp\left(-(\frac{\beta}{N+1} - 1)k_N \cdot t\right) \text{Lip}(f) \tag{1.5}$$

*for all $f \in Lip(\overline{\Sigma}_N)$, where $k_N > 0$ is a universal constant depending only on $N$.*

In the proof of Theorem 1.1 for the more difficult case $\beta < (N + 1)$ we use some localization arguments to exploit the geometric symmetries of the problem. A crucial ingredient for this approach is the following ('Markov-')uniqueness result for the operator $(L^N, C^2_{Neu})$ which is interesting in its own right.

**Proposition 1.2.** *For $\beta < 2(N + 1)$, there is at most one symmetric strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on $L^2(\Sigma_N, q_N)$ whose generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ extends $(L^N, C^2_{Neu})$.*

Hence, together with Theorem 1.1 we can state the following existence and uniqueness result.

**Corollary 1.3.** *The formal system of Skorokhod SDEs defines via the associated martingale problem a unique diffusion process which can be started everywhere on the (closed) simplex $\overline{\Sigma}_N$.*

As for path regularity we obtain the following characterization.

**Theorem 1.4.** *For any starting point $x \in \overline{\Sigma}_N$, $(X^x_\cdot) \in \overline{\Sigma}_N \subset \mathbb{R}^N$ is a Euclidean semimartingale if and only if $\beta/(N + 1) \geq 1$.*

In particular we obtain that a Skorokhod decomposition of the process $X^N$ is impossible if $\beta$ is small enough. This is in sharp contrast to all aforementioned previous works. Moreover, again due to the uniqueness assertion of Proposition 1.2 the following negative result holds true.

**Corollary 1.5.** *If $\beta/(N + 1) < 1$, the system of equations (1.3),(1.4) is ill-posed, i.e. it admits no solution in the the sense of Itô calculus.*

Theorems 1.1 and 1.4 generalize the corresponding classical results for the familiy of standard real Bessel processes, which are proved in a completely different manner, c.f. [6, 17].

*Strategy of proof.* While the proof of Theorem 1.4 consists of a straightforward application of a regularity criterion for Dirichlet processes by Fukushima [13], the proof of Theorem 1.1 is more involved, and entirely different methods are used in the two respective cases $\beta < N + 1$ and $\beta \geq N + 1$. Both cases are non-trivial from an analytic point of view due to the degenerating coefficients and the Neumann boundary condition.

By comparison the case $\beta \geq (N + 1)$ is much easier since $q_N$ is then log-concave. Using a recent powerful result by Ambrosio, Savaré and Zambotti [2] on the stability of gradient flows for the relative entropy functional on Wasserstein the contraction estimate (1.5) is established by smooth approximation and coupling.

For $\beta < (N + 1)$ the measure $q_N$ is no longer log-concave and to prove the Feller property we proceed in four main steps. The first crucial observation is that the reference measure $q_N$ can locally be extended to a measure $\hat{q}$ on the full Euclidean space which lies in the Muckenhoupt class $\mathcal{A}_2$, allowing for a rich potential theory. In particular the Poincaré inequality and doubling condition hold which imply via heat kernel estimates the regularity of the induced process on $\mathbb{R}^N$. The second step is the probabilistic piece of the localization, which corresponds to stopping the $\mathbb{R}^N$-valued process at domain boundaries where the measure $\hat{q}$ is 'tame' and to show that the stopped process is again Feller. To this end we employ a version of the Wiener test for degenerate elliptic diffusions by Fabes, Jerrison and Kenig [9]. The third step is to use the reflection symmetry of the problem which allows to treat the Neumann boundary condition indeed via a reflection of the extended $\mathbb{R}^N$-valued process. The fourth step is to establish the Markov uniqueness which is crucial in order to justify the identification of the processes after localization, and here we shall again depend on the nice potential theory available in the Muckenhoupt class.

The partial resemblance of our proof for $\beta < (N+1)$ to the classical work by Bass and Hsu [5] is no surprise. However, we did not find any similar work in the probability literature where potential theory for Muckenhoupt weights was used so extensively as in our case. We think that the general strategy outlined here may be useful also in other, less specific situations.

## 2 Dirichlet Form and Integration by Parts Formula

We start with the rigorous construction of $(X_.^N)$, which departs from the symmetrizing measure $q_n$ on $\Sigma_N$ defined above in (1.1). Note that one can identify $L^p(\Sigma_N, q_N)$ with $L^p(\overline{\Sigma}_N, q_N)$, $p \geq 1$. Throughout the paper $q_N$ denotes both the measure and its Lebesque density, i.e. $q_N(A) = \int_A q_N(x)\,dx$ for all measurable $A \subseteq \overline{\Sigma}_N$.

For all $\beta > 0$, $N \in \mathbb{N}$, the measure $q_N$ satisfies the 'Hamza condition', because it has a strictly positive density with locally integrable inverse, c.f. e.g. [1]. This implies that the form $\mathcal{E}^N(f, f)$ with domain $f \in C^\infty(\overline{\Sigma}_N)$ is closable on $L^2(\Sigma_N, q_N)$. The $L^2(\Sigma_N, q_N)$-closure defines a local regular Dirichlet form, still denoted by $\mathcal{E}^N$. General Dirichlet form theory asserts the existence of a Hunt diffusion $(X_.^N)$ associated with $\mathcal{E}^N$ which can be started in $q_N$-almost all $x \in \overline{\Sigma}_N$ and which is understood as a generalized solution of the system (1.3),(1.4). This identification is justified by the fact that any semi-martingale solution solves via Itô's formula the martingale problem for the operator $(L^N, D(L^N))$, defined in (1.2).

The following integration by parts formula for $q_N$ can be easily verified (e.g. by approximation from corresponding integrals over increasing sub-domains $\Sigma_N^\epsilon \subset \Sigma_N$).

**Proposition 2.1.** *Let $u \in C^1(\overline{\Sigma}_N)$ and $\xi = (\xi^1, \ldots, \xi^N)$ be a vector field in $C^1(\overline{\Sigma}_N, \mathbb{R}^N)$ satisfying $\langle \xi, \nu \rangle = 0$ on $\partial \Sigma_N$, where $\nu$ denotes the outward normal field of $\partial \Sigma_N$. Then,*

$$\int_{\Sigma_N} \langle \nabla u, \xi \rangle q_N(dx) = - \int_{\Sigma_N} u \left[ \mathrm{div}(\xi) + (\tfrac{\beta}{N+1} - 1) \sum_{i=1}^N \xi^i \left( \frac{1}{x^i - x^{i-1}} - \frac{1}{x^{i+1} - x^i} \right) \right] q_N(dx).$$

**Remark 2.2.** Let $u \in C^1(\overline{\Sigma}_N)$ and $\xi$ be a vector field of the the form $\xi = w\vec{\varphi}$ with $w \in C^1(\overline{\Sigma}_N)$ and $\vec{\varphi}(x) = (\varphi(x^1), \ldots, \varphi(x^N))$, $\varphi \in C^\infty([0,1])$ and $\langle \vec{\varphi}, \nu \rangle = 0$ on $\partial \Sigma_N$, in particular $\varphi_{|\partial[0,1]} = 0$. Then, the integration by parts formula above reads

$$\int_{\Sigma_N} \langle \nabla u, \xi \rangle q_N(dx) = - \int_{\Sigma_N} u \left[ w V_{N,\varphi}^\beta + \langle \nabla w, \vec{\varphi} \rangle \right] q_N(dx),$$

where

$$V_{N,\varphi}^\beta(x^1, \ldots, x^N) := (\frac{\beta}{N+1} - 1) \sum_{i=0}^N \frac{\varphi(x^{i+1}) - \varphi(x^i)}{x^{i+1} - x^i} + \sum_{i=1}^N \varphi'(x^i).$$

Let $C_{Neu}^2 = \{ f \in C^2(\overline{\Sigma}_N) : \langle \nabla f, \nu \rangle = 0 \text{ on } \partial \Sigma_N \}$ as above with $\nu$ still denoting the outer normal field on $\partial \Sigma_N$. Then, for any $f \in C^1(\overline{\Sigma}_N)$ and $g \in C_{Neu}^2$ we apply the integration by parts formula in Proposition 2.1 for $\xi = \nabla g$ to obtain

$$\mathcal{E}^N(f, g) = - \int_{\Sigma_N} f\, L^N g\, q_N(dx).$$

Moreover, it is easy to show that

$$\left| \mathcal{E}^N(f, g) \right| \leq C \, \|f\|_{L^2(\Sigma_N, q_N)}, \qquad \forall f \in D(\mathcal{E}^N).$$

In particular, $C_{Neu}^2$ is contained in the domain of the generator $\mathcal{L}$ associated with $\mathcal{E}^N$ and $L^N f = \mathcal{L} f$ for all $f \in C_{Neu}^2$.

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# 3 Feller Property

This section is devoted to the proof of Theorem 1.1, where we treat the cases $\beta < (N+1)$ and $\beta \geq (N+1)$ separately. Both cases are not trivial from an analytic perspective due to the combination of degenerate coefficients and the Neumann boundary condition. In fact we could not find any general result in the PDE literature which contains the current model. Our proof for the case $\beta < (N+1)$ avoids an explicit treatment of the Neumann boundary condition via a reflection argument. In the case $\beta \geq (N+1)$ we use a powerful stability property of log-concave measures.

## 3.1 Case $\beta < (N+1)$.

For $\beta < (N+1)$ the measure $q_N$ is no longer log-concave. However, the proof below extends also to all cases when $\beta < 2(N+1)$. Let

$$\Omega_N := \Omega_N^\delta := \overline{\Sigma}_N \cap \{x \in \mathbb{R}^N : x_N \leq 1 - \delta\},$$

for some positive small $\delta < [2(N+2)]^{-1}$ and the weight function

$$\hat{q}(x) := \hat{q}_{N,\delta}(x) := \begin{cases} q_N(x) & \text{if } x \in \Omega_N, \\ \frac{1}{Z_\beta} \delta^{\frac{\beta}{N+1}-1} \prod_{i=1}^N (x^i - x^{i-1})^{\frac{\beta}{N+1}-1} & \text{if } x \in S_1 \backslash \Omega_N, \end{cases} \tag{3.1}$$

where

$$S_1 := \{x \in \mathbb{R}^N : 0 \leq x^1 \leq x^2 \leq \ldots \leq x^N\}.$$

We want to extend the weight function $\hat{q}$ to the whole $\mathbb{R}^N$. To do this we introduce the mapping

$$T : \mathbb{R}^N \to S_1 \quad x \mapsto (|x^{(1)}|, \ldots, |x^{(N)}|), \tag{3.2}$$

where $(1).\ldots,(N)$ denotes the permutation of $1,\ldots,N$ such that

$$|x^{(1)}| \leq \ldots \leq |x^{(N)}|.$$

The extension of $\hat{q}$ on $\mathbb{R}^N$ is now defined via $\hat{q}(x) = \hat{q}(Tx)$, $x \in \mathbb{R}^N$. Again we will also denote by $\hat{q}$ the induced measure on $\mathbb{R}^N$. Consider the $L^2(\mathbb{R}^N, \hat{q})$-closure of

$$\hat{\mathcal{E}}^{N,a}(f,f) = \int_{\mathbb{R}^N} \langle a\nabla f, a\nabla f(x) \rangle \, \hat{q}(dx), \quad f \in C_c^\infty(\mathbb{R}^N) \tag{3.3}$$

still denoted by $\hat{\mathcal{E}}^{N,a}$ for a measurable field $x \mapsto a(x) \in \mathbb{R}^{N \times N}$ on $\mathbb{R}^N$ satisfying

$$\frac{1}{c} \cdot E_N \leq a(x)^t \cdot a(x) \leq c \cdot E_N \tag{3.4}$$

in the sense of non-negative definite matrices. Let $(Y_t)_{t \geq 0} = (Y_t^{N,a})_{t \geq 0}$ be the associated symmetric Hunt process on $\mathbb{R}^N$, starting from the invariant distribution $\hat{q}$. Finally, we denote by $(Q_t)_t$ the transition semigroup of $Y$.

### 3.1.1 Feller Properties of $Y$

Let $C_0(\mathbb{R}^N)$ be the space of continuous functions on $\mathbb{R}^N$ vanishing at infinity.

**Proposition 3.1.** *For $2(N+1) > \beta$ and a matrix $a$ satisfying (3.4), $Y^{N,a}$ is a Feller process o $\mathbb{R}^N$, i.e.*

   *i) for every $t > 0$ and every $f \in C_0(\mathbb{R}^N)$ we have $Q_t f \in C_0(\mathbb{R}^N)$,*

   *ii) for every $f \in C_0(\mathbb{R}^N)$, $\lim_{t\downarrow 0} Q_t f = f$ pointwise in $\mathbb{R}^N$.*

*Moreover, $Q_t f \in C(\mathbb{R}^N)$ for every $t > 0$ and every $f \in L^2(\mathbb{R}^N, \hat{q})$.*

**Remark 3.2.** It is well known that i) and ii) even imply that $\lim_{t\downarrow 0} \|Q_t f - f\|_\infty = 0$ for each $f \in C_0(\mathbb{R}^N)$. Moreover, the following version of the strong Markov property holds. Let $T$ be a stopping time with $T \leq t_0$ a.s. for some $t_0 > 0$. Then, for each $f \in L^2(\mathbb{R}^N, \hat{q})$

$$E\left[f(Y_{t_0}) \,|\, \mathcal{F}_T\right] = E_{Y_T}\left[f(Y_{t_0-T})\right],$$

with $(\mathcal{F}_t)_{t\geq 0}$ denoting the natural filtration of $Y$.

*Proof of Proposition 3.1.* ii) follows directly by path-continuity and dominated convergence. i) as well as the additional statement follow from the analytic regularity theory of symmetric diffusions, see [19], in particular Theorem 3.5 and Proposition 3.1 and Corollary 4.2, provided the following two conditions are fulfilled:

- The measure $\hat{q}$ is doubling, i.e. there exists a constant $C'$, such that for all Euclidean balls $B_R \subset B_{2R}$
$$\hat{q}(B_{2R}) \leq C'\hat{q}(B_R).$$

- $\hat{\mathcal{E}}$ satisfies a uniform local Poincaré inequality, i.e. there is a constant $C' > 0$ such that
$$\int_{B_R} |f - (f)_{B_R}|^2 d\hat{q} \leq C'R^2 \int_{B_R} |\nabla f|^2 d\hat{q},$$
for all Euclidean balls $B_R$ and $f \in \mathcal{D}(\hat{\mathcal{E}})$, where $(f)_{B_R}$ denotes the integral $\frac{1}{\hat{q}(B_R)}\int_{B_R} f d\hat{q}$.

Both conditions are verified once we have proven that the weight function $\hat{q}$ is contained in the Muckenhoupt class $\mathcal{A}_2$, which will be done in Lemma 3.3 below. Indeed, the doubling property follows immediately from the Muckenhoupt condition (see e.g. [23] or [21]) and for the proof of the Poincaré inequality see Theorem 1.5 in [10]. $\qquad\square$

**Lemma 3.3.** *For $N$ such that $\beta < 2(N+1)$, we have $\hat{q} \in \mathcal{A}_2$, i.e. there exists a positive constant $C = C(N,\delta)$ such that for every Euclidean ball $B_R$*

$$\frac{1}{|B_R|}\int_{B_R} \hat{q}\, dx\, \frac{1}{|B_R|}\int_{B_R} \hat{q}^{-1}\, dx \leq C,$$

*where $|B_R|$ denotes the Lebesgue measure of the ball $B_R$.*

It suffices to prove the Muckenhoupt condition for the weight function $\tilde{q}$ defined by

$$\tilde{q}(x) := \prod_{i=1}^{N}(x^i - x^{i-1})^{\beta/(N+1)-1}, \quad x \in S_1, \tag{3.5}$$

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and $\tilde{q}(x) := \tilde{q}(Tx)$ if $x \in \mathbb{R}^N$, since there exist positive constants $C_1$ and $C_2$ depending on $\delta$ and $N$ such that

$$C_1\, \tilde{q}(x) \leq \hat{q}(x) \leq C_2\, \tilde{q}(x), \qquad \forall x \in \mathbb{R}^N. \tag{3.6}$$

Note that $(1 - x_N)^{\frac{\beta}{N+1}-1}$ is uniformly bounded and bounded away from zero on $\Omega_N$.
Below $P_R(m)$ denotes the parallelepiped in $\mathbb{R}^N$ with basis point $m = (m_1, \ldots, m_N)$, which is spanned by the vectors $v_i = \sum_{j=i}^N e_j$, $i = 1, \ldots, N$, normalized to the length $R$, where $(e_j)_{j=1,\ldots,N}$ is the canonical basis in $\mathbb{R}^N$. Then, $P_R(m)$ can also be written as

$$P_R(m) = m + \left\{ x \in \mathbb{R}^N : x^1 \in \left[0, \tfrac{R}{\sqrt{N}}\right], x^2 \in \left[x^1, x^1 + \tfrac{R}{\sqrt{N-1}}\right], \ldots, x^N \in \left[x^{N-1}, x^{N-1} + R\right] \right\}.$$

**Lemma 3.4.** *Let $B_R$ be an arbitrary Euclidean ball in $\mathbb{R}^N$. Then, there exists a positive constant $C$ only depending on $N$ and a parallelepiped $P_{lR}(m) \subset S_1$ with $l > 0$ independent of $B_R$ such that*

$$\int_{B_R} \tilde{q}^{\pm 1} dx \leq C \int_{P_{lR}(m)} \tilde{q}^{\pm 1} dx.$$

*Proof.* We denote by $(S_n)$, $n = 1, \ldots, 2^N \cdot N!$, the subsets of $\mathbb{R}^N$ taking the form

$$S_n = \{x \in \mathbb{R}^N : (s_1 \pi_1(x), \ldots, s_N \pi_N(x)) \in S_1\},$$

where $s_i \in \{-1, 1\}$, $i = 1, \ldots, N$, and $\pi(x) = (\pi_1(x), \ldots, \pi_N(x))$ is a permutation of the components of $x$. Note that $\mathbb{R}^N = \bigcup_n S_n$ and the intersections of the sets $S_n$ have zero Lebesgue measure.

Let now $B_R(M)$ be an arbitrary Euclidean ball with radius $R$ centered in $M \in \mathbb{R}^N$. We first consider the case $|M| \leq 2R$. Then, obviously $B_R(M) \subset B_{4R}(0)$ and $B_{4R}(0) \cap S_1 \subset P_{4R}(0)$. Hence,

$$\int_{B_R(M)} \tilde{q}^{\pm 1} dx \leq \int_{B_{4R}(0)} \tilde{q}^{\pm 1} dx = \sum_n \int_{B_{4R}(0) \cap S_n} \tilde{q}^{\pm 1} dx.$$

By the definition of $\tilde{q}$ we have

$$\int_{B_{4R}(0) \cap S_n} \tilde{q}^{\pm 1} dx = \int_{B_{4R}(0) \cap S_1} \tilde{q}^{\pm 1} dx$$

for all $n \in \{1, \ldots, 2^N \cdot N!\}$. Thus,

$$\int_{B_R(M)} \tilde{q}^{\pm 1} dx \leq 2^N \cdot N! \int_{B_{4R}(0) \cap S_1} \tilde{q}^{\pm 1} dx \leq C \int_{P_{4R}(0)} \tilde{q}^{\pm 1} dx.$$

Suppose now that $|M| > 2R$. Let $n_0$ be such that $M \in S_{n_0}$. Then, by construction of $\tilde{q}$ we have

$$\int_{B_R(M) \cap S_n} \tilde{q}^{\pm 1} dx \leq \int_{B_R(M) \cap S_{n_0}} \tilde{q}^{\pm 1} dx$$

for all $n = 1, \ldots, 2^N \cdot N!$. Hence,

$$\int_{B_R(M)} \tilde{q}^{\pm 1} dx = \sum_n \int_{B_R(M) \cap S_n} \tilde{q}^{\pm 1} dx \leq 2^N \cdot N! \int_{B_R(M) \cap S_{n_0}} \tilde{q}^{\pm 1} dx.$$

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Set $K := T(B_R(M) \cap S_{n_0}) \subset S_1$ with $T$ defined as above. Then, it is clear by definition of $\tilde{q}$ that

$$\int_{B_R(M) \cap S_{n_0}} \tilde{q}^{\pm 1} dx = \int_K \tilde{q}^{\pm 1} dx$$

and we get

$$\int_{B_R(M)} \tilde{q}^{\pm 1} dx \leq C \int_K \tilde{q}^{\pm 1} dx.$$

Finally, we choose a parallelepiped $P_{2R}(m)$ such that $K \subseteq P_{2R}(m) \subset S_1$, which completes the proof. $\square$

*Proof of Lemma 3.3.* We prove the Muckenhoupt condition for $\tilde{q}$. Recall that we have assumed $\beta < 2(N+1)$. In the following the symbol $C$ denotes a positive constant depending on $N$ and $\delta$ with possibly changing its value from one occurence to another. Using Lemma 3.4 we have

$$\frac{1}{|B_R|^2} \int_{B_R} \tilde{q}\, dx \int_{B_R} \tilde{q}^{-1} dx \leq CR^{-2N} \int_{P_{lR}(m)} \tilde{q}\, dx \int_{P_{lR}(m)} \tilde{q}^{-1} dx$$

$$= CR^{-2N} \int_{P_{lR}(m)} \prod_{i=1}^{N} (x^i - x^{i-1})^{\beta/(N+1)-1} dx$$

$$\times \int_{P_{lR}(m)} \prod_{i=1}^{N} (x^i - x^{i-1})^{-(\beta/(N+1)-1)} dx.$$

By the change of variables $y_i = x^i - x^{i-1}$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{|B_R|^2} \int_{B_R} \tilde{q}\, dx \int_{B_R} \tilde{q}^{-1} dx \leq CR^{-2N} \prod_{i=1}^{N} \int_{\tilde{m}_i}^{\tilde{n}_i} y_i^{\frac{\beta}{N+1}-1} dy_i \int_{\tilde{m}_i}^{\tilde{n}_i} y_i^{-(\frac{\beta}{N+1}-1)} dy_i,$$

where we have set $\tilde{m}_i := m_i - m_{i-1}$ with $m_0 := 0$ and $\tilde{n}_i := \tilde{m}_i + \frac{lR}{\sqrt{N+1-i}}$ for abbreviation. Recall that in one dimension the weight function $x \mapsto |x|^\eta$ on $\mathbb{R}$ is contained in $\mathcal{A}_2$ if $\eta \in (-1, 1)$ (see p. 229 and p. 236 in [21]). Hence, we get for every $i \in \{1, \ldots, N\}$

$$\int_{\tilde{m}_i}^{\tilde{n}_i} y_i^{\frac{\beta}{N+1}-1} dy_i \int_{\tilde{m}_i}^{\tilde{n}_i} y_i^{-(\frac{\beta}{N+1}-1)} dy_i \leq C \left| \frac{lR}{\sqrt{N+1-i}} \right|^2,$$

and the result follows.

$\square$

### 3.1.2 Feller Properties of $Y$ inside a Box

Let $E := E^\delta := \{x \in \mathbb{R}^N : \|x\|_\infty < 1 - 2\delta\}$ be a box in $\mathbb{R}^N$ centered in the origin and $\mathcal{B}(E)$ be the Borel $\sigma$-field on $E$. We denote by

$$\tau_E := \inf\{t > 0 : Y_t \in E^c\}$$

the first exit time of $E$. This subsection is devoted to the proof of the Feller properties for the stopped process $Y_t^E := Y_t^{N,a,E} := Y_{t \wedge \tau_E}^{N,a}$, whose transition semigroup is given by

$$Q_t^E f(x) = E^x[f(Y_{t \wedge \tau_E})] = E^x\left[ f(Y_t)\mathbb{1}_{\{t < \tau_E\}} + f(Y_{\tau_E})\mathbb{1}_{\{t \geq \tau_E\}} \right], \qquad t > 0, \, x \in \bar{E},$$

for every bounded $f$ on $\bar{E}$. In order to prove the Feller property we will follow essentially the proof of Theorem 13.5 in [7]. It is shown there that the Feller properties are preserved, if the domain is regular in the following sense.

**Proposition 3.5.** *The domain $E$ is regular, i.e. for every $z \in \partial E$ we have $P^z[\tau_E = 0] = 1$.*

**Remark 3.6.** $z \in \partial E$ is regular point in the sense of the definition given in Proposition 3.5 if and only if for every continuous function $f$ on $\partial E$

$$\lim_{E \ni x \to z} E^x[f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}}] = f(z).$$

For the Brownian case we refer to Theorem 2.12 in [14]. The arguments are robust, but it is required that under $P^x$ the probability of the event that the exit time of a ball centered at $x$ does not exceed $t$ is arbitrary small uniformly in $x$ for a suitable chosen small $t > 0$. In our situation this property is ensured by [18], p. 330.

The proof of this proposition will be based on the following Wiener test established by Fabes, Jerison and Kenig, c.f. [9, Thorem 5.1].

**Theorem 3.7.** *Let $B_{R_0}(0)$ be a large ball centered in zero such that $E \subset B_{R_0/4}(0)$. Then, a point $z \in \partial E$ is regular if and only if*

*a)* $\int_0^{R_0} \frac{s^2}{\hat{q}(B_s(z))} \frac{ds}{s} < \infty$, *or*

*b)* $\int_0^{R_0} \mathrm{cap}(K_\rho) \frac{\rho^2}{\hat{q}(B_\rho(z))} \frac{d\rho}{\rho} = \infty$,

*where $K_\rho(z) := (B_{R_0}(0) \backslash E) \cap B_\rho(z)$ and $\mathrm{cap}$ denotes the capacity associated with the Dirichlet form $\mathcal{E}^{N,a}$.*

In the sequel we will use the following notation

$$d(x) := \left|\{i \in \{1, \ldots, N\} : x^i = x^{i-1}\}\right|, \qquad x \in \mathbb{R}^N,$$

again with the convention $x^0 := 0$. In order to prove the regularity of $E$ we start with a preparing lemma.

**Lemma 3.8.** *Let $y \in \mathbb{R}^N$ be arbitrary.*

i) *There exists a positive $r_0 = r_0(y)$ such that for all balls $B_r(y)$, $0 < r \leq r_0$, there exists a parallelepiped $P_{lr}(\bar{y})$ contained in $S_1$ with $d(\bar{y}) = d(y)$, a positive constant $C$ and $l > 0$, such that*

$$\hat{q}(B_r(y)) \leq C\hat{q}(P_{lr}(\bar{y})).$$

ii) *For all balls $B_r(y)$ there exists parallelepiped $P_{lr}(\bar{y})$ contained in $S_1$ with $d(\bar{y}) = d(y)$, for some $l > 0$ such that $\hat{q}(P_{lr}(\bar{y})) \leq \hat{q}(B_r(y))$.*

iii) *For all $y$ we have $C_1 r^{h(y)} \leq \hat{q}(B_r(y))$ for all $r$ and $\hat{q}(B_r(y)) \leq C_2 r^{h(y)}$ for all $r \leq r_0(y)$ for some positive constants $C_1$ and $C_2$ depending on $y$, where*

$$h(y) := N - d(y) + \frac{\beta}{N+1} d(y).$$

*Proof.* Obviously, due to (3.6) it suffices to prove the lemma with $\hat{q}$ replaced by $\tilde{q}$ defined in (3.5).

i) The case $y = 0$ is clear. For $y \neq 0$ we choose $r_0$ such that $2r_0 \leq |y|$ and let $n_0$ be such that $y \in S_{n_0}$. Consider now an arbitrary ball $B_r(y)$ with $r \leq r_0$ and let $K = T(S_{n_0} \cap B_r(y))$ be the subset of $S_1$ constructed in the second part of the proof of Lemma 3.4. Then, possibly after

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choosing a smaller $r_0$ we find a parallelepiped and a positive constant $l$ such that $K \subset P_{lr}(\bar{y}) \subset S_1$ with $d(\bar{y}) = d(y)$ and we obtain i).

ii) Let $K = T(S_{n_0} \cap B_r(y))$ be defined as in i). Then, clearly $\tilde{q}(B_r(y)) \geq \tilde{q}(K)$. Thus, we can choose $\bar{y} = Ty$ and $l$ independent of $r$ such that $P_{lr}(\bar{y}) \subseteq K$ and ii) follows.

iii) We proceed similar as in the proof of Lemma 3.3. For some parallelepiped $P_{lr}(\bar{y})$ with $d(\bar{y}) = d(y)$ we use a change of variables to obtain

$$\tilde{q}(P_{lr}(\bar{y})) = \int_{P_{lr}(\bar{y})} \prod_{i=1}^{N} (x^i - x^{i-1})^{\frac{\beta}{N+1}-1} dx = \prod_{i=1}^{N} \int_{\tilde{y}_i}^{\tilde{y}_i + c_i r} z_i^{\frac{\beta}{N+1}-1} dz_i,$$

where $\tilde{y}_i = \bar{y}_i - \bar{y}_{i-1}$, $\bar{y}_0 := 0$ and $c_i := \frac{l}{\sqrt{N+1-i}}$ . Note that $d(\bar{y}) = d(y)$ is the number of components of $\tilde{y}$ which are equal to zero. Using the mean value theorem we obtain that

$$C_1 r^{h(y)} \leq \hat{q}(P_{lr}(\bar{y})) \leq C_2 r^{h(y)}$$

for some positive constants $C_1$ and $C_2$ depending on $y$, so that iii) follows from i) and ii). $\square$

*Proof of Proposition 3.5.* Let $z \in \partial E$ be fixed and $R_0$ be as in the statement of Theorem 3.7. Setting $h'(z) := 1 - h(z)$ let us first consider the case $h'(z) > -1$. Then, using Lemma 3.8 iii) we have that

$$\int_0^{R_0} \frac{s^2}{\hat{q}(B_s(z))} \frac{ds}{s} \leq C \int_0^{r_0} s^{h'(z)} ds + \frac{1}{2\hat{q}(B_{r_0}(z))} (R_0^2 - r_0^2) < \infty,$$

with $r_0 = r_0(z)$ as above in Lemma 3.8 iii). Thus, the criterion a) in Theorem 3.7 applies and the regularity of $z$ follows. The case $h'(z) \leq -1$ is more difficult. Combining Lemma 3.1 in [10] and Theorem 3.3 in [10] and using Lemma 3.8 iii) we get the following estimate for the capacity of small balls:

$$\mathrm{cap}(B_r(y)) \simeq \left( \int_r^{R_0} \frac{s^2}{\hat{q}(B_s(y))} \frac{ds}{s} \right)^{-1} \geq C \left( \int_r^{R_0} s^{h'(y)} ds \right)^{-1}. \tag{3.7}$$

Recall the definition of the set $K_\rho(z)$ in Theorem 3.7. Clearly, for every $\rho$ sufficiently small there exists a ball $B_{\rho/2}(\hat{z})$ with $\hat{z}$ depending on $\rho$ and with $d(\hat{z}) = d(z)$ such that $B_{\rho/2}(\hat{z}) \subset K_\rho(z)$. Let now $r_0$ be such that Lemma 3.8 iii) and (3.7) hold for every ball $B_\rho(z)$, $\rho \leq r_0$. Then, we obtain in the case $h'(z) < -1$

$$\int_0^{R_0} \mathrm{cap}(K_\rho) \frac{\rho^2}{\hat{q}(B_\rho(z))} \frac{d\rho}{\rho} \geq C \int_0^{r_0} \mathrm{cap}(B_{\rho/2}(\hat{z})) \rho^{h'(z)} d\rho \geq C \int_0^{r_0} \left( \int_{\rho/2}^{R_0} s^{h'(\hat{z})} ds \right)^{-1} \rho^{h'(z)} d\rho$$

$$= C \, (h'(z) + 1) \int_0^{r_0/2} \frac{\rho^{h'(z)}}{R_0^{h'(y)+1} - \rho^{h'(y)+1}} d\rho$$

$$= C \int_0^{r_0/2} \left( - \log(|R_0^{h'(y)+1} - \rho^{h'(y)+1}|) \right)' d\rho = \infty.$$

Finally, if $h'(z) = -1$ we get by an analogous procedure

$$\int_0^{R_0} \mathrm{cap}(K_\rho) \frac{\rho^2}{\hat{q}(B_\rho(z))} \frac{d\rho}{\rho} \geq C \int_0^{r_0/2} \frac{1}{\rho(\log R_0 - \log \rho)} d\rho$$

$$= C \int_0^{r_0/2} \left( - \log(\log R_0 - \log \rho) \right)' d\rho = \infty.$$

Hence, applying the criterion b) of Theorem 3.7 completes the proof. $\square$

**Proposition 3.9.** $Y^E$ *is a Feller process, i.e.*

  *i) for every $t > 0$ and every $f \in C(\bar{E})$ we have $Q_t^E f \in C(\bar{E})$,*

  *ii) for every $f \in C(\bar{E})$, $\lim_{t \downarrow 0} Q_t^E f = f$ pointwise in $\bar{E}$.*

The statement is classical and can be found e.g. in Theorem 13.5 in [7]. For readability we repeat the argument here. We shall need the following lemma.

**Lemma 3.10.** *For any compact set $K \subset E$ we have*

$$\limsup_{t \downarrow 0} \; \sup_{x \in K} P^x[\tau_E \leq t] = 0.$$

*Proof.* We need to show that for any $\delta > 0$ there exists a $t_0 > 0$ such that

$$\inf_{x \in K} P^x[\tau_E \geq t_0] \geq 1 - \delta. \tag{3.8}$$

Consider a bounded function $f \in C_0(\mathbb{R}^N)$ such that $0 \leq f \leq 1$, $f = 1$ on $K$ and $f = 0$ on the complement of $E$. Let now $t_0$ be such that $\sup_{t \leq t_0} \|Q_t f - f\|_\infty < \delta/2$ (cf. Remark 3.2). Then,

$$E^x[f(Y_{t_0})] \leq P^x[\tau_E \geq t_0] + E^x[f(Y_{t_0}) \, \mathbb{1}_{\{\tau_E < t_0\}}].$$

For $x \in K$ the left hand side is equal to

$$Q_{t_0} f(x) = 1 + Q_{t_0} f(x) - f(x) \geq 1 - \sup_{t \leq t_0} \|Q_t f - f\|_\infty \geq 1 - \frac{\delta}{2}.$$

On the other hand, using the strong Markov property (cf. again Remark 3.2) and the fact that $f(Y_{\tau_E}^N) = 0$ we have

$$E^x[f(Y_{t_0}) \, \mathbb{1}_{\{\tau_E < t_0\}}] = E^x\left[E^x[f(Y_{t_0}) | \mathcal{F}_{\tau_E}] \, \mathbb{1}_{\{\tau_E < t_0\}}\right] = E^x\left[Q_{t_0 - \tau_E} f(Y_{\tau_E}) \, \mathbb{1}_{\{\tau_E < t_0\}}\right]$$

$$= E^x\left[(Q_{t_0 - \tau_E} f(Y_{\tau_E}) - f(Y_{\tau_E})) \, \mathbb{1}_{\{\tau_E < t_0\}}\right] \leq E^x\left[\sup_{t \leq t_0} \|Q_t f - f\|_\infty \, \mathbb{1}_{\{\tau_E < t_0\}}\right]$$

$$\leq \frac{\delta}{2},$$

and (3.8) follows. $\qquad \square$

*Proof of Proposition 3.9.* Let $t > 0$ and $f \in C(\bar{E})$. Then, by the semigroup property of $(Q_t^E)$ we have for $0 < s < t$

$$Q_t^E f(x) = E^x\left[\psi_s(Y_{s \wedge \tau_E})\right],$$

where

$$\psi_s(x) = Q_{t-s}^E f(x) = E^x\left[f(Y_{(t-s) \wedge \tau_E})\right], \qquad x \in \bar{E}.$$

Then, $\psi_s$ can be extended to a function in $L^2(\mathbb{R}^N, \hat{q})$ and by Proposition 3.1 we have $Q_s \psi_s \in C_b(\mathbb{R}^N)$. Since

$$\left|Q_t^E f(x) - Q_s \psi_s(x)\right| = |E^x[\psi_s(Y_{s \wedge \tau_E}) - \psi_s(Y_s)]| \leq 2 \|\psi_s\| \, P^x[\tau_E \leq s] \leq 2 \|f\| \, P^x[\tau_E \leq s]$$

and since the right hand side converges to zero uniformly in $x$ on every compact subset of $E$ by Lemma 3.10, we conclude that $Q_t^E f \in C_b(E)$, i.e. $Q_t^E f$ is continuous in the interior of $E$. In order to show i) it suffices to verify continuity at the boundary. Since $E$ is regular, we have

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obviously $Q_t^E f = f$ on $\partial E$. By Lemma 13.1 in [7] we have upper semicontinuity of the mapping $x \mapsto P^x[t < \tau_E]$. Hence, we obtain for $z \in \partial E$,

$$\limsup_{x \to z} P^x[t < \tau_E] \leq P^z[t < \tau_E] = 0,$$

where we have used the regularity of $E$ in Proposition 3.5. Thus, for every $x \in E$

$$
\begin{aligned}
\left| Q_t^E f(x) - f(z) \right| &\leq \left| Q_t^E f(x) - E^x[f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}}] \right| + \left| E^x[f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}}] - f(z) \right| \\
&\leq \left| E^x \left[ f(Y_t) \mathbb{1}_{\{t < \tau_E\}} + f(Y_{\tau_E}) \mathbb{1}_{\{t \geq \tau_E\}} - f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}} \right] \right| \\
&\quad + \left| E^x[f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}}] - f(z) \right| \\
&\leq \|f\|_\infty P^x[t < \tau_E] + \left| E^x \left[ f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}} \left( \mathbb{1}_{\{t \geq \tau_E\}} - 1 \right) \right] \right| \\
&\leq 2\|f\|_\infty P^x[t < \tau_E] + \left| E^x[f(Y_{\tau_E}) \mathbb{1}_{\{\tau_E < \infty\}}] - f(z) \right|,
\end{aligned}
$$

where we have used the fact that the event $\{\tau_E < \infty\}$ is included in $\{t \geq \tau_E\}$. Since $z$ is a regular point, the second term tends to zero as $x \to z$, $x \in E$, cf. Remark 3.6. Hence,

$$\lim_{E \ni x \to z} Q_t^E f(x) = f(z)$$

and property i) is proven.

To prove ii), we extend $f$ to a function in $C_0(\mathbb{R}^N)$, i.e. we may deduce from Proposition 3.1 that $\lim_{t \downarrow 0} Q_t f(x) = f(x)$ for every $x \in E$. Furthermore, we have for every $x \in E$

$$|Q_t^E f(x) - Q_t f(x)| \leq P^x[t \leq \tau_E] \|f\|_\infty,$$

and $P^x[\tau_E = 0] = 0$, since $E$ is open and $Y$ has continuous paths. Hence,

$$\lim_{t \downarrow 0} Q_t^E f(x) = \lim_{t \downarrow 0} Q_t f(x) = f(x).$$

This gives pointwise convergence on $E$. Since $Q_t^E f = f$ on $\partial E$ by regularity, the convergence on $\partial E$ is trivial. $\qquad \square$

### 3.1.3 Feller Properties of $X^N$

In this section we will finally prove the Feller property for $X^N$ stated in Theorem 1.1. To this end we construct a Feller process $\tilde{X}$ taking values in $\overline{\Sigma}_N$ and in a second step we will identify this process with $X^N$.

In analogy to the definition of $\Omega_N$ above we set

$$\Omega_i := \Omega_i^\delta := \{x \in \overline{\Sigma}_N : x^{i+1} - x^i \geq \delta\}, \qquad i = 0, \ldots, N. \tag{3.9}$$

and moreover

$$A^i := \partial \Omega_i^{2\delta} \backslash \partial \Sigma_N = \{x \in \overline{\Sigma}_N : x^{i+1} - x^i = 2\delta\}.$$

Notice that we can choose $\delta$ so small that $\overline{\Sigma}_N = \bigcup_{i=1}^N \Omega_i^{2\delta}$. Furthermore, we define the mappings $H_i$, $i = 0, \ldots, N$, by

$$H_i(x) := \left( x^1, \ldots, x^i, 1 - (x^N - x^i), 1 - (x^{N-1} - x^i), \ldots, 1 - (x^{i+1} - x^i) \right), \qquad x \in \overline{\Sigma}_N.$$

Notice that for every $i$, $H_i$ maps $\Omega_N$ on $\Omega_i$ and vice versa. In particular, $H_i \circ H_i = \mathrm{id}$ and $H_N$ is the identity on $\Omega_N$. Let $T : \bar{E} \to \Omega_N^{2\delta} \subset \Omega_N$ be defined as in (3.2). Let $Y^i$ denote the

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$\mathbb{R}^N$-valued Feller process induced from the form (3.3) with $a = a_i$, where the matrix valued function $a$ is defined $\hat{q}$-almost everywhere by

$$a_i(x) := \begin{cases} DH_i^t & \text{for } x \in \mathring{S}_1 \\ DH_i^t \cdot (DT_{|S_n}^{-1})^t & \text{for } x \in \mathring{S}_n, \, n \in \{2, \cdots 2^N \cdot N!\}, \end{cases} \tag{3.10}$$

such that condition (3.4) is clearly satisfied (note that $DH_i = DH_i^{-1}$). In particular, $a_i$ is constant on every $S_n$. Moreover, setting $\rho_i := H_i \circ T$, clearly $a_i = (D\rho_i^{-1})^t$.

**Remark 3.11.** Analogously to Proposition 2.1 one can establish the following integration by parts formula associated to $\mathcal{E}^{N, a_i}$. Let $u \in C^1(\mathbb{R}^N)$ and $\xi$ a continuous vector field on $\mathbb{R}^N$ such that $\operatorname{supp}\xi \subseteq \{x \in \mathbb{R}^N : \|x\|_\infty < 1 - \delta\}$, $\xi$ is continuously differentiable in the interior of each $S_n$ and for every $n$ $\xi$ satisfies the boundary condition $\langle a_i^t \cdot a_i \cdot \xi, \nu_n \rangle = 0$ on $\partial S_n$, $\nu_n$ denoting the outward normal field of $\partial S_n$. Then,

$$\int_{\mathbb{R}^N} \langle a_i \nabla u, a_i \xi \rangle \hat{q}(dx) = - \int_{\mathbb{R}^N} u \left[ \operatorname{div}(a_i^t \cdot a_i \cdot \xi) + \langle a_i^t \cdot a_i \cdot \xi, \nabla \log \hat{q} \rangle \right] \hat{q}(dx).$$

Thus, every smooth function $g$ such that $\xi = \nabla g$ satisfies the above conditions is contained in the domain of the generator $L^i$ associated to $\mathcal{E}^{N, a_i}$ and on $\mathbb{R}^N \setminus \bigcup_n \partial S_n$ we have

$$L^i g = \operatorname{div}(a_i^t \cdot a_i \cdot \nabla g) + \langle a_i^t \cdot a_i \cdot \nabla g, \nabla \log \hat{q} \rangle.$$

Next we define the $\Omega_i^{2\delta}$-valued process $\tilde{X}^i$ by

$$\tilde{X}_t^i = \rho_i(Y_t^{i,E}) = H_i \circ T(Y_t^{i,E}), \qquad t \geq 0.$$

The semigroup of $\tilde{X}^i$ will be denoted by $(\tilde{P}_t^i)_{t \geq 0}$, i.e. $\tilde{P}_t^i f(x) = E^x[f(\tilde{X}_t^i)]$.

**Lemma 3.12.** *For every $i$, $\tilde{X}^i$ is Markovian.*

*Proof.* Since $H_i$ is an injective mapping for every $i$, it suffices to show that the process $T(Y^{i,E})$ is Markovian. Moreover, since $T(Y_t^{i,E}) = T(Y_{t \wedge \tau_E}^i) = (T(Y_\cdot^i))_{t \wedge \tau_E}$ it is enough to prove the Markov property for the process $T(Y^i)$, which is implied e.g. by the condition that for any Borel set $A \subseteq \overline{\Sigma}_N$

$$P^x[Y_t^i \in T^{-1}(A)] = P^y[Y_t^i \in T^{-1}(A)] \qquad \text{whenever } T(x) = T(y). \tag{3.11}$$

Now the choice of $\hat{q}$ and the metric $a_i$ imply for any Borel set $A \subseteq \overline{\Sigma}_N$ condition (3.11) is satisfied. To see this, let $\{\sigma_k \mid k = 1, \cdots, N + N(N-1)/2\}$ be the collection of line-reflections in $\mathbb{R}^N$ with respect to either one of the coordinate axes $\{\lambda e_i, \lambda \in \mathbb{R}\}$ or a diagonal $\{\lambda(e_j + e_k)\}$, then for $x, y \in \mathbb{R}^N$ with $T(x) = T(y)$ there exists a finite sequence $\sigma_{k_1}, \cdots \sigma_{k_l}$ such that $\tau(x) := \sigma_{k_1} \circ \sigma_{k_2} \cdots \circ \sigma_{k_l}(x) = y$. Now each of the reflections $\sigma_i$ preserves the Dirichlet form (3.3) when $a$ is chosen as in (3.10), such that the processes $\tau(Y^{i,x})$ and $Y^{i,y}$ are equal in distribution. Moreover, $\tau(T^{-1}(A)) = T^{-1}(A)$, from which (3.11) is obtained. $\square$

**Lemma 3.13.** *For each $f \in C_{Neu}^2$ the process $t \to f(\tilde{X}_t^i) - \int_0^t L^N f(\tilde{X}_s^i) \, ds$ is a martingale w.r.t. to the filtration generated by $\tilde{X}_\cdot^i$.*

*Proof.* Similar to Proposition 2.1 one checks for $f \in C_{Neu}^2 \cap C_c(\Omega_i)$ that the function $f_i$ on $\mathbb{R}^N$, which is defined by $f_i = f \circ H_i \circ T = f \circ \rho_i$ on the set $\{x \in \mathbb{R}^N : \|x\|_\infty < 1 - \delta\}$ and $f_i = 0$ on the complement this set, belongs to the domain of the generator $L^i$ of the Dirichlet form

(3.3) with $a = a_i$ as in (3.10) (cf. Remark 3.11). Hence the process $f_i(Y^i_\cdot) - \int_0^\cdot L^i f_i(Y^i_s)ds$ is a martingale w.r.t. to the filtration generated by $Y^i$ and thus also $f_i(Y^{i,E}_\cdot) - \int_0^\cdot L^i f_i(Y^{i,E}_s)ds$ due to the optional sampling theorem. Obviously in the last statement the function $f$ can be modified outside of $\Omega_i^{2\delta}$, i.e. it holds also for $f \in C^2_{Neu}$. Moreover, $f_i(Y^{i,E}_\cdot) = f(\tilde{X}^i_\cdot)$ and $L^i f_i = (L^N f) \circ \rho_i$ on $\bar{E}$. Thus, $f(\tilde{X}^i_\cdot) - \int_0^\cdot L^N f(\tilde{X}^i_s)ds$ is a martingale w.r.t. the filtration generated by $Y^i_\cdot$ and adapted to the filtration generated by $\tilde{X}^i_\cdot = H_i \circ T(Y^{i,E}_\cdot)$ which establishes the claim. $\qquad\square$

**Proposition 3.14.** *For every $i$, $\tilde{X}^i$ is a Feller process, more precisely*

   *i) for every $t > 0$ and every $f \in C(\Omega_i^{2\delta})$ we have $\tilde{P}^i_t f \in C(\Omega_i^{2\delta})$,*

   *ii) for every $f \in C(\Omega_i^{2\delta})$, $\lim_{t\downarrow 0} \tilde{P}^i_t f = f$ pointwise in $\Omega_i^{2\delta}$.*

*Proof.* Since obviously for every continuous $f$ on $\Omega_i^{2\delta}$

$$\tilde{P}^i_t f(x) = E^x[f(\tilde{X}^i_t)] = E^x[f(H_i \circ T(Y^{i,E}_t))] = Q^E_t(f \circ H_i \circ T)(x), \qquad t > 0,$$

the result follows from Proposition 3.9 and the continuity of the mappings $H_i$ and $T$. $\qquad\square$

Next we define the process $\tilde{X}$ with state space $\overline{\Sigma}_N$ as follows: Let $q_N$ be the initial distribution of $\tilde{X}$, i.e. $\tilde{X}_0 \sim q_N$. Choose $i_1 \in \{0, \ldots N\}$ such that $\tilde{X}_0 \in \Omega_{i_1}^{2\delta}$ and $\mathrm{dist}(\tilde{X}_0, A^{i_1}) = \max_i \mathrm{dist}(\tilde{X}_0, A^i)$. We set $\tilde{X}_t = \tilde{X}^{i_1}_t$ for $0 \le t \le T_1$, where $T_1$ denotes the first hitting time of $A^{i_1}$, i.e. on $[0, T_1]$ the process behaves according to $\tilde{P}^{i_1}$. Choose now $i_2 \in \{0, \ldots N\}$ such that $\tilde{X}_{T_1} \in \Omega_{i_2}^{2\delta}$ and $\mathrm{dist}(\tilde{X}_{T_1}, A^{i_2}) = \max_i \mathrm{dist}(\tilde{X}_{T_1}, A^i)$. At time $T_1$ the process starts afresh from $\tilde{X}_{T_1}$ according to $(\tilde{P}^{i_2}_t)$ up to the first time $T_2$ after $T_1$, when it hits $A^{i_2}$. This procedure is repeated forever.

**Proposition 3.15.** *$\tilde{X}$ is a Feller process.*

*Proof.* Let $(\tilde{P}_t)$ denote the semigroup associated to $\tilde{X}$. For $f \in C(\overline{\Sigma}_N)$, we need to show that $\tilde{P}_t f \in C(\overline{\Sigma}_N)$ for every $t > 0$. Let us first show that $\tilde{P}_{T_n} f \in C(\overline{\Sigma}_N)$ for every $n$ using an induction argument. For an arbitrary $x \in \overline{\Sigma}_N$, choose $i_1$ as above depending on $x$ such that $\tilde{P}_{T_1} f(x) = \tilde{P}^{i_1}_{T_1} f(x)$. Since $\tilde{P}^{i_1}_{T_1} f \in C(\Omega_{i_1}^{2\delta})$, we conclude that $\tilde{P}_{T_1} f$ is continuous in $x$ for every $x$. For arbitrary $n$ and $x \in \overline{\Sigma}_N$ we have by the strong Markov property

$$\tilde{P}_{T_{n+1}} f(x) = E^x[f(\tilde{X}_{T_{n+1}})] = E^x\left[E_{\tilde{X}_{T_N}}[f(\tilde{X}^{i_n}_{T_{n+1}-T_n})]\right] = E^x\left[\tilde{P}^{i_n}_{T_{n+1}-T_n} f(\tilde{X}_{T_N})\right]$$
$$= \tilde{P}_{T_n}(P^{i_n}_{T_{n+1}-T_n} f)(x)$$

and since $P^{i_n}_{T_{n+1}-T_n} f$ can be extended to a continuous function on $\overline{\Sigma}_N$, we get $\tilde{P}_{T_{n+1}} f \in C(\overline{\Sigma}_N)$ by the induction assumption.

Similarly, one can show that for every $n$ the mapping $x \mapsto E^x[f(\tilde{X}_t)\mathbb{1}_{\{t \in (T_n, T_{n+1}]\}}]$ is continuous. Finally, for every $x \in \overline{\Sigma}_N$

$$\left| \tilde{P}_t f(x) - \sum_{k=0}^{n-1} E^x\left[f(\tilde{X}_t)\mathbb{1}_{\{t \in (T_k, T_{k+1}]\}}\right]\right| = \left| E^x\left[f(\tilde{X}_t)\mathbb{1}_{\{t > T_n\}}\right]\right| \le \|f\|_\infty P^x[t > T_n]$$

and since $T_n \nearrow \infty$ $P^x$-a.s. locally uniformly in $x$ as $n$ tends to infinity, the claim follows. $\qquad\square$

The proof of Theorem 1.1 is complete, once we have shown that the processes $\tilde{X}$ and the original process $X^N$ have the same law.

**Lemma 3.16.** *For $x \in \overline{\Sigma}_N$ the process $(\tilde{X}_\cdot^x)$ obtained from conditioning $\tilde{X}$ to start in $x$, solves the martingale problem for the operator $(L^N, C^2_{Neu})$ as in (1.2) and starting distribution $\delta_x$.*

*Proof.* Let $\{T_k\}$ be the sequence of strictly increasing stopping times introduced in the construction of the process $\tilde{X}$. Then for $s < t$

$$f(\tilde{X}_t) - f(\tilde{X}_s) - \int_s^t L^N f(\tilde{X}_\sigma) d\sigma = \sum_k f(\tilde{X}_{(T_{k+1}\vee s)\wedge t}) - f(\tilde{X}_{(T_k\vee s)\wedge t}) - \int_{(T_k\vee s)\wedge t}^{(T_{k+1}\vee s)\wedge t} L^N f(\tilde{X}_\sigma) d\sigma.$$

Hence

$$\mathbb{E}\big(f(\tilde{X}_t) - \int_0^t L^N f(\tilde{X}_\sigma) d\sigma \,\big|\, \mathcal{F}_s\big) = f(\tilde{X}_s) - \int_0^s L^N f(\tilde{X}_\sigma) d\sigma$$

$$+ \sum_k \mathbb{E}\big(f(\tilde{X}_{(T_{k+1}\vee s)\wedge t}) - f(\tilde{X}_{(T_k\vee s)\wedge t}) - \int_{(T_k\vee s)\wedge t}^{(T_{k+1}\vee s)\wedge t} L^N f(\tilde{X}_\sigma) d\sigma \,\big|\, \mathcal{F}_s\big).$$

Using the strong Markov property of the Feller process $\tilde{X}_\cdot$ and its pathwise decomposition into pieces of $\{\tilde{X}^i\}$-trajectories one obtains

$$\mathbb{E}\big(f(\tilde{X}_{(T_{k+1}\vee s)\wedge t}) - f(\tilde{X}_{(T_k\vee s)\wedge t}) - \int_{(T_k\vee s)\wedge t}^{(T_{k+1}\vee s)\wedge t} L^N f(\tilde{X}_\sigma) d\sigma \,\big|\, \mathcal{F}_s\big)$$

$$= \mathbb{E}\big[\mathbb{E}_{\tilde{X}_{(T_k\vee s)\wedge t}}\big(f(\tilde{X}^{i_k}_{\tau\wedge(t-s)}) - f(\tilde{X}^{i_k}_0) - \int_0^{\tau\wedge(t-s)} L^N f(\tilde{X}^{i_k}_\sigma) d\sigma\big) \,\big|\, \mathcal{F}_s\big], \qquad (3.12)$$

where, by construction of $\tilde{X}_\cdot$, $\tau$ ist the hitting time of the the the set $A_{i_k}$ for which $\text{dist}(A_{i_k}, \tilde{X}_{T_k}) = \max_i \text{dist}(A_i, \tilde{X}_{T_k})$. Lemma 3.13 implies that the inner expectation in (3.12) is zero. $\square$

The last ingredient for our proof of Theorem 1.1 is the identification of the processes $\tilde{X}_\cdot$ with $X_\cdot$. Since both are Markovian and solve the martingale problem for the operator $(L^N, C^2_{Neu})$ it suffices to show that the martingale problem admits at most one Markovian solution. Clearly, any such solution induces a symmetric sub-Markovian semigroup on $L^2(\Sigma_N, q_N)$ whose generator extends $(L^N, C^2_{Neu})$. Hence it is enough to establish the following so-called Markov uniqueness property of $(L^N, C^2_{Neu})$, cf. [8, Definition 1.2], stated in Proposition 1.2.

*Proof of Proposition 1.2.* Let $H^{1,2}(\Sigma_N, q_N)$ (resp. '$H_0^{1,2}(\Sigma_N, q_N)$' in the notation of [8]) denote the closure of $C^2_{Neu}$ w.r.t. to the norm $\|f\|_1 = (\|f\|^2_{L^2(\Sigma_N,q_N)} + \|\nabla f\|^2_{L^2(\Sigma_N,q_N)})^{1/2}$ and let $W^{1,2}$ be the Hilbert space of $L^2(\Sigma_N, q_N)$-functions $f$ whose distributional derivative $Df$ is in $L^2(\Sigma_N, q_N)$, equipped with the norm $\|f\|_1 = (\|f\|^2_{L^2(\Sigma_N,q_N)} + \|Df\|^2_{L^2(\Sigma_N,q_N)})^{1/2}$. Clearly, the quadratic form $Q(f,f) = \langle Df, Df\rangle_{L^2(\Sigma_N,q_N)}$, $\mathcal{D}(Q) = W^{1,2}(\Sigma_N, q_N)$, is a Dirichlet form on $L^2(\Sigma_N, q_N)$. Hence we may use the basic criterion for Markov uniqueness [8, Corollary 3.2], according to which $(L, C^2_{Neu})$ is Markov unique if $H^{1,2} = W^{1,2}$.

To prove the latter it obviously suffices to prove that $H^{1,2}$ is dense in $W^{1,2}$. Again we proceed by localization as follows. For fixed $\delta > 0$ let $\Omega_i^\delta = \Omega_i$ denote the subsets defined in (3.9), then $\{\Omega_i^{3\delta}\}$ constitutes a relatively open covering of $\overline{\Sigma}_N$ for $\delta$ small enough. Let $\{\eta_i\}_{i=0,\cdots,N}$ and $\{\chi_i\}_{i=0,\cdots,N}$ be two smooth partitions of unity on $\overline{\Sigma}_N$ such that $\eta_i = 1$ on $\Omega_i^{3\delta}$ and $\text{supp}(\eta_i) \subset \Omega_i^{2\delta}$ and $\chi_i = 1$ on $\Omega_i^{2\delta}$ and $\text{supp}(\chi_i) \subset \Omega_i^\delta$ respectively. Hence writing $f \in W^{1,2}(\Sigma_N, q_N)$ as $f = \sum_i f_i$ with

$f_i = \eta_i \cdot f \in W^{1,2}(\Sigma_N, q_N)$ it suffices to prove that $f_i \in H^{1,2}(\Sigma_N, q_N)$. We show first that $f_i$ can be approximated w.r.t. $\|.\|_1$ by functions which are smooth up to boundary and secondly that such functions can be approximated in $\|.\|_1$ by smooth Neumann functions. We give details for the case $i = N$ only, the other cases can be treated almost the same way by using the maps $H_i$.

Let $g_N = f \cdot \chi_N \in W^{1,2}(\Sigma_N, q_N)$, then the restriction of $g_N$ to $\Omega_N^\delta$ belongs to the space $W^{1,2}(\Omega_N^\delta, q_N) = W^{1,2}(\Omega_N^\delta, \hat{q})$, where $\hat{q}$ denotes the modification of $q_N$ according to (3.1). Due to Lemma 3.3 the extension $\hat{q}(x) = \hat{q}(T(x))$, $x \in \mathbb{R}^N$, lies in the Muckenhoupt class $\mathcal{A}_2$. Further note that $W^{1,2}(\Omega_N^\delta, \hat{q}) \subset W^{1,2}(\Omega_N^\delta, dx) = H^{1,2}(\Omega_N^\delta, dx)$ if $\beta \leq N + 1$, and $W^{1,2}(\Omega_N^\delta, \hat{q}) \subset W^{1,1}(\Omega_N^\delta, dx) = H^{1,1}(\Omega_N^\delta, dx)$ by the Hölder inequality if $N + 1 < \beta < 2(N+1)$, such that $g_N$ has well defined boundary values in $L^1(\partial \Omega_N, dx)$. Hence we may conclude that the extension $\hat{g}_N(x) = g_N(T(X))$, $x \in \mathbb{R}^N$ defines a weakly differentiable function on $\mathbb{R}^N$ with $\|\hat{g}_N\|_{W^{1,2}(\mathbb{R}^N, \hat{q})} = 2^N \cdot N! \; \|g_N\|_{W^{1,2}(\Omega_N, \hat{q})}$. By [15, Theorem 2.5] the mollification with the standard mollifier yields an approximating sequence $\{u_l\}_l$ of smooth functions $u_l \in C_0^\infty(\mathbb{R}^N)$ of $g_N$ in the weighted Sobolev spaces $H^{1,2}(\mathbb{R}^N, \hat{q})$. Extending each $u_l \cdot \eta_N$ by zero in $\overline{\Sigma}_N \setminus \Omega_N^\delta$ we obtain a sequence of $C^\infty(\overline{\Sigma}_N)$-functions $\{u_l \cdot \eta_N\}_l$ which converges to $g_N \cdot \eta_N = f_N$ in $H^{1,2}(\Sigma_N, q_N)$. This finishes the first step. In the second step we thus may assume w.l.o.g. that $g_N$ is smooth up to the boundary of $\Sigma_N$. In particular, $g_N$ is globally Lipschitz. Since $q_N$ is integrable on $\Sigma_N$ we may modify $f_N$ close to the boundary to obtain a Lipschitz function $\tilde{f}_N$ which satisfies the Neumann condition and which is close to $f_N$ in $\|.\|_1$. (Take, e.g. $\tilde{f}_N(x) = f(\pi(x))$, where $\pi(x)$ is the projection of $x$ into the set $\Sigma_N^\epsilon = \{x \in \Sigma_N, |\mathrm{dist}(x, \partial \Sigma_N) \geq \epsilon\}$ for small $\epsilon > 0$.) We may now proceed as in step one to obtain an approximation of $\tilde{f}_N$ by smooth functions w.r.t. $\|.\|_1$, where we note that neither extension by reflection through the map $T$ nor the standard mollification in [15] of the extended $\tilde{f}_N$ destroys the Neuman boundary condition. $\square$

Hence we arrive at the following statement which implies the first statement of Theorem 1.1 for the cases when $\beta < 2(N + 1)$.

**Corollary 3.17.** *For quasi-every $x \in \overline{\Sigma}_N$, the the processes $\tilde{X}_\cdot^x$ and $X_\cdot^x$ are equal in law. In particular, $X$ is a Feller process on $\overline{\Sigma}_N$.*

## 3.2 The case $\beta \geq (N + 1)$.

The estimate (1.5) looks (also for the case $\beta < N + 1$) like a straightforward application of the Bakry-Emery $\Gamma_2$-calculus. However, the complete Bakry-Emery criterion requires the $\Gamma_2$-condition on an algebra of functions which is dense in the domain of the generator $\mathcal{L}$ of $\mathcal{E}$ w.r.t. the graph norm. The verification of the latter typically leads back to elliptic regularity theory for $\mathcal{L}$ which we want to avoid. Instead our argument below is based on a recent powerful result by Ambrosio, Savare and Zambotti [2] on the stability of reversible processes with log-concave invariant measures.

*Proof of* (1.5). By abuse of notation let $q_N$ be the extension of the measure $q_N$ to $\mathbb{R}^N$ given by

$$q_N(A) = q_N(A \cap \Sigma_N) = \frac{1}{Z_\beta} \int_{A \cap \{V < \infty\}} e^{-(\frac{\beta}{N+1} - 1)V(x)} dx$$

for any Borel set $A \subset \mathbb{R}^N$, where $V : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$,

$$V(x) = \begin{cases} -\sum_{i=1}^{N+1} \ln(x_i - x_{i-1}) & x \in \Sigma_N \\ \infty & x \in \mathbb{R}^N \setminus \Sigma_N \end{cases}$$

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is a lower semicontinuous and convex function on $\mathbb{R}^N$. In fact, by elementary calculations, for $x \in \Sigma_N$ with $\rho_i = \rho_i(x) = 1/(x_i - x_{i-1})^2$, $i = 1, \ldots, N+1$,

$$\langle \xi, \nabla^2 V(x) \xi \rangle = \rho_1 \xi_1^2 + \sum_{i=1}^{N-1} \rho_{i+1}(\xi_{i+1} - \xi_i)^2 + \rho_{N+1}\xi_N^2 \geq k_N |\xi|^2, \quad \forall \xi \in \mathbb{R}^N,$$

where $k_N > 0$ denotes the smallest eigenvalue of the strictly elliptic matrix $A = (2\delta_{ij} - \delta_{1,|i-j|}) \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$. Hence the measure $q_N$ is log-concave in the sense of [2]. (Note that according to ibid. Theorem 1.2.b. the Feller property of $X$. on $\overline{\Sigma}_N$ is automatically implied.) Let $\overline{x} \in \Sigma_N$ denote the barycenter of the simplex $\Sigma_N$. For $\epsilon > 0$ let

$$V_\epsilon(x) = \begin{cases} V((1-\epsilon)x + \epsilon\overline{x})) & x \in \overline{\Sigma}_N \\ \infty & x \in \mathbb{R}^N \setminus \overline{\Sigma}_N \end{cases}$$

and define the measure $q_N^\epsilon$ on $\mathbb{R}^N$ by

$$q_N^\epsilon(A) = \frac{1}{Z_\beta} \int_{A \cap \Sigma_N} e^{-(\frac{\beta}{N+1}-1)V_\epsilon(x)} dx.$$

Since $V_\epsilon \in C^\infty(\Sigma_N)$ and the boundary of $\Sigma_N$ is piecewise smooth this process can also be constructed by the corresponding Skorokhod SDE. Moreover, the classical coupling be reflection method can be applied, c.f. [25]. Taking expectations in [25, eq. (2.5)] (note that $I = 0$ in our case) and using the strict convexity of $V_\epsilon$ together with Gronwall's lemma this yields the estimate

$$\mathbb{E}_C(\overline{X}_t^{\epsilon,1}, \overline{X}_t^{\epsilon,2}) \leq e^{-(\frac{\beta}{N+1}-1)(1-\epsilon)^2 k_N t} d(x, y),$$

where $C$ denotes the law of the coupling process $(\overline{X}_t^{\epsilon,1}, \overline{X}_t^{\epsilon,2})_{t \geq 0}$, starting in $(x, y)$. In particular the following estimate in the $L^1$-Wasserstein distance $d_1$ for the heat kernel of $(X^\epsilon)$ is obtained

$$d_1(P_t^\epsilon(x, \cdot), P_t^\epsilon(y, \cdot)) \leq e^{-(\frac{\beta}{N+1}-1)(1-\epsilon)^2 k_N t} d(x, y).$$

Since $q_N^\epsilon \to q_N$ weakly, we may now invoke Theorem 6.1. of [2] in order to pass to the limit for $\epsilon \to 0$ in the left hand side above, using also the continuity of the $L^1$- w.r.t the $L^2$-Wassserstein metric. Hence we arrive at

$$d_1(P_t(x, \cdot), P_t(y, \cdot)) \leq e^{-(\frac{\beta}{N+1}-1)(1-\epsilon)^2 k_N t} d(x, y),$$

for all $\epsilon > 0$ small enough, thus also for $\epsilon = 0$. Via Kantorovich duality this implies

$$|P_t f(x) - P_t f(y)| \leq e^{-(\frac{\beta}{N+1}-1)k_N t} \mathrm{Lip}(f) \, d(x, y),$$

for or all $f \in \mathrm{Lip}(\Sigma_N)$ and $x, y \in \Sigma_N$, which is the claim. $\qquad\square$

## 4 Semi-Martingale Properties

In this final section we prove the semi-martingale properties of $X^N$ stated in Theorem 1.4. To that aim we establish the semi-martingale properties for the symmetric process $X^N$ started in equilibrium, which imply the semi-martingale properties to hold for quasi-every starting point $x \in \overline{\Sigma}_N$ and by the Feller properties proven in the last section for every starting point $x \in \overline{\Sigma}_N$. In order to establish the semi-martingale properties of the stationary process, we shall use the following criterion established by Fukushima in [13]. For every open set $G \subset \overline{\Sigma}_N$ we set

$$\mathcal{C}_G := \{u \in D(\mathcal{E}^N) \cap C(\overline{\Sigma}_N) : \mathrm{supp}(u) \subset G\}.$$

**Theorem 4.1.** *For $u \in D(\mathcal{E}^N)$ the additive functional $u(X_t^N) - u(X_0^N)$ is a semi-martingale if and only if one of the following (equivalent) conditions holds:*

    *i) For any relatively compact open set $G \subset \overline{\Sigma}_N$, there is a positive constant $C_G$ such that*

$$|\mathcal{E}^N(u, v)| \leq C_G \|v\|_\infty, \qquad \forall v \in \mathcal{C}_G. \tag{4.1}$$

    *ii) There exists a signed Radon measure $\nu$ on $\overline{\Sigma}_N$ charging no set of zero capacity such that*

$$\mathcal{E}^N(u, v) = -\int_{\overline{\Sigma}_N} v(x) \, \nu(dx), \qquad \forall v \in C(\overline{\Sigma}_N) \cap \mathcal{D}(\mathcal{E}^N). \tag{4.2}$$

*Proof.* See Theorem 6.3 in [13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Theorem 4.2.** *Let $X^N$ be a symmetric diffusion on $\overline{\Sigma}_N$ associated with the Dirichlet form $\mathcal{E}^N$, then $X^N$ is an $\mathbb{R}^N$-valued semi-martingale if and only if $\beta/(N+1) \geq 1$.*

*Proof.* Since the semi-martingale property for $\mathbb{R}^N$-valued diffusions is defined componentwise, we shall apply Fukushima's criterion for $u(x) = x^i$, $i = 1, \ldots, N$.

Let us first consider the case where $\beta' := \beta/(N+1) > 1$. Then, for a relatively compact open set $G \subset \overline{\Sigma}_N$ and $v \in \mathcal{C}_G$,

$$
\begin{aligned}
\mathcal{E}^N(u, v) &= \int_{\Sigma_N} \frac{\partial}{\partial x^i} v(x) \, q_N(dx) \\
&= \frac{1}{Z_\beta} \int_0^1 dx^1 \int_{x^1}^1 dx^2 \cdots \int_{x^{i-1}}^1 dx^{i+1} \int_{x^{i+1}}^1 dx^{i+2} \cdots \int_{x^{N-1}}^1 dx^N \prod_{\substack{j=0 \\ j \neq i-1, i}}^N (x^{j+1} - x^j)^{\beta'-1} \\
&\quad \times \int_{x^{i-1}}^{x^{i+1}} \frac{\partial}{\partial x^i} v(x) \, (x^i - x^{i-1})^{\beta'-1} (x^{i+1} - x^i)^{\beta'-1} \, dx^i.
\end{aligned}
$$

Since $\beta' > 1$, we obtain by integration by parts

$$
\begin{aligned}
&\int_{x^{i-1}}^{x^{i+1}} \frac{\partial}{\partial x^i} v(x) \, (x^i - x^{i-1})^{\beta'-1} (x^{i+1} - x^i)^{\beta'-1} \, dx^i \\
&= -(\beta' - 1) \int_{x^{i-1}}^{x^{i+1}} v(x) \left[ (x^i - x^{i-1})^{\beta'-2} (x^{i+1} - x^i)^{\beta'-1} - (x^i - x^{i-1})^{\beta'-1} (x^{i+1} - x^i)^{\beta'-2} \right] dx^i
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left| \int_{x^{i-1}}^{x^{i+1}} \frac{\partial}{\partial x^i} v(x) \, (x^i - x^{i-1})^{\beta'-1} (x^{i+1} - x^i)^{\beta'-1} \, dx^i \right| \\
&\leq (\beta' - 1) \|v\|_\infty \left( \int_{x^{i-1}}^{x^{i+1}} (x^i - x^{i-1})^{\beta'-2} \, dx^i + \int_{x^{i-1}}^{x^{i+1}} (x^{i+1} - x^i)^{\beta'-2} \, dx^i \right) \\
&\leq 2(\beta' - 1) \|v\|_\infty \int_0^1 r^{\beta'-2} dr \leq C \|v\|_\infty,
\end{aligned}
$$

and we obtain that condition (4.1) holds. For $\beta' = 1$ the measure $q_N$ coincides with the normalized Lebesgue measure on $\overline{\Sigma}_N$, condition (4.1) follows directly.

Let now $\beta' < 1$ and let us assume that $u(X_t^N) - u(X_0^N)$ is a semi-martingale. Then, there exists a signed Radon measure $\nu$ on $\overline{\Sigma}_N$ satisfying (4.2). Let $\nu = \nu_1 - \nu_2$ be the Jordan decomposition of $\nu$, i.e. $\nu_1$ and $\nu_2$ are positive Radon measures. By the abobe calculations we have for each relatively compact open set $G \subset \overline{\Sigma}_N$ and for all $v \in \mathcal{C}_G$

$$
\mathcal{E}^N(u,v) = -\frac{1}{Z_\beta}(\beta' - 1) \int_G v(x) \prod_{\substack{j=0 \\ j \neq i-1,\, i}}^{N} (x^{j+1} - x^j)^{\beta'-1}
$$
$$
\times \left[ (x^i - x^{i-1})^{\beta'-2}(x^{i+1} - x^i)^{\beta'-1} - (x^i - x^{i-1})^{\beta'-1}(x^{i+1} - x^i)^{\beta'-2} \right] dx.
$$

Hence, we obtain for the Jordan decomposition $\nu = \nu_1 - \nu_2$ that

$$
\nu_1(G) = \frac{1}{Z_\beta}(1 - \beta') \int_G (x^{i+1} - x^i)^{\beta'-2} \prod_{\substack{j=0 \\ j \neq i}}^{N}(x^{j+1} - x^j)^{\beta'-1} dx
$$

$$
\nu_2(G) = \frac{1}{Z_\beta}(1 - \beta') \int_G (x^i - x^{i-1})^{\beta'-2} \prod_{\substack{j=0 \\ j \neq i-1}}^{N}(x^{j+1} - x^j)^{\beta'-1} dx.
$$

Set $\partial\Sigma_N^j := \{x \in \partial\Sigma_N : x^j = x^{j+1}\}$, $j = 0, \ldots, N$, and let for some $x_0 \in \partial\Sigma^i$ and $r > 0$, $A := x_0 + [-r, r]^N \cap \overline{\Sigma}_N$ be such that $\mathrm{dist}(A, \partial\Sigma_N^j) > 0$ for all $j \neq i$. Furthermore, let $(A_n)_n$ be a sequence of compact subsets of $A$ such that $A_n \uparrow A$ and $\mathrm{dist}(A_n, \partial\Sigma_N^i) > 0$ for every $n$.
By the inner regularity of the Radon measures $\nu_1$ and $\nu_2$ we have $v_1(A) = \lim_n \nu_1(A_n)$ and $v_2(A) = \lim_n \nu_2(A_n)$. Since $\beta' - 2 < -1$, we get by the choice of $A$ that $\nu_1(A) = \infty$, while $\nu_2(A) < \infty$, which contradicts the local finiteness of $\nu$ and $\nu_1$, respectively. $\qquad\square$

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